

# THE PEGG-BARNETT FORMALISM AND COVARIANT PHASE OBSERVABLES

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ABSTRACT. We compare the Pegg-Barnett (PB) formalism with the covariant phase observable approach to the problem of quantum phase and show that PB-formalism gives essentially the same results as the canonical (covariant) phase observable. We also show that PB-formalism can be extended to cover all covariant phase observables including the covariant phase observable arising from the angle margin of the Husimi  $Q$ -function.

## 1. INTRODUCTION

There are at least three different approaches to the problem of quantum phase. First, one may quantize some appropriate classical dynamical variable, e.g. the phase angle of two dimensional phase space, using some quantization rule to get a self-adjoint phase operator. The most widely used quantization rule is the Cahill-Glauber  $s$ -parametrized [1] quantization and especially one of its special cases, the Wigner-Weyl quantization (see, e.g. [2, 3]). Second, in the Pegg-Barnett (PB) formalism, one may calculate the phase properties of a single mode electromagnetic field using a sequence of self-adjoint operators acting on finite subspaces of the infinite dimensional Hilbert space of the single mode system (see, e.g. [4]). The third way to approach the phase problem is to extend the mathematical representation of the concept of quantum observable from a self-adjoint operator to a normalized positive operator measure (POM, for short) and assume that any quantum phase observable is a phase shift covariant POM with the interval  $[0, 2\pi)$  as the range of its possible measurement outcomes.

It has been shown [5] that the  $s$ -quantized phase angle operators arise from phase shift covariant generalized operator measures. They are covariant POMs when the quantization parameter  $s$  is suitably chosen but, for example, in the case of the Wigner-Weyl quantized phase angle this is not the case [5, 6]. Still, the structure of the  $s$ -quantized phase angle operators is similar to the first moment operators of phase shift covariant POMs. The PB-formalism can also be embedded in the covariant approach and in this article we show that the PB-formalism can be extended to cover the whole covariant theory. Before doing this, we recall some basic physical properties which are important in the theory of quantum phase.

As it appears from the abundant literature, the main principles used to define a phase observable are the following:

- P0. The range of values of a phase observable is the interval  $[0, 2\pi)$ .
- P1. The number observable generates phase shifts.
- P2. The phase is completely random in the number states.
- P3. A phase observable generates number shifts.

Within the POM approach, these postulates can be incorporated in a logical fashion as follows: P0 and P1 lead to define a phase observable as a phase shift covariant POM based on  $[0, 2\pi)$ . Any such operator measure fulfils P2, but P0 and P2 do not imply P1. P3 requires that a phase observable generates number shifts, which forces a phase shift covariant POM to be strong. Then P0, P1, and P3 imply that the phase observable is the canonical (covariant) phase observable (modulo unitary equivalence). Thus, the canonical phase observable is (up

to unitary equivalence), the only phase observable which has all the four physically relevant properties P0–P4.

The structure of this article is as follows. In Sec. 2 we give our basic notations and definitions. In Sec. 3 we show that phase shift covariance is a natural condition for phase observables which describe phase measurements in coherent states. In Sec. 4 we introduce covariant phase observables and, finally, in Sec. 5 we show the connections between the PB-formalism and the covariant phase observable approach.

## 2. BASIC NOTATIONS

Let  $\mathcal{H}$  be a complex separable Hilbert space, and fix an orthonormal basis  $\{|n\rangle \in \mathcal{H} \mid n = 0, 1, 2, \dots\}$  on it. We call it the number basis. With respect to that we define the lowering operator  $a := \sum_{n=0}^{\infty} \sqrt{n+1} |n\rangle \langle n+1|$ , its adjoint the raising operator  $a^* := \sum_{n=0}^{\infty} \sqrt{n+1} |n+1\rangle \langle n|$ , and the number operator  $N := a^*a = \sum_{n=0}^{\infty} n |n\rangle \langle n|$ , with their usual domains. The unitary operators  $R(\theta) := e^{i\theta N}$ ,  $\theta \in \mathbb{R}$ , are called phase shifters. If  $|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$ ,  $z \in \mathbb{C}$ , is a coherent state, then  $R(\theta)|z\rangle = |ze^{i\theta}\rangle$  for all  $\theta \in \mathbb{R}$ , showing the fundamental fact that the number operator generates phase shifts in coherent states.

Let  $\mathcal{L}(\mathcal{H})$  denote the set of bounded operators on  $\mathcal{H}$ , and  $\mathcal{B}(\Omega)$  the  $\sigma$ -algebra of the Borel subsets of a set  $\Omega \subseteq \mathbb{C}$ . Let  $E : \mathcal{B}(\Omega) \rightarrow \mathcal{L}(\mathcal{H})$  be a normalized positive operator measure. We recall that a map  $E : \mathcal{B}(\Omega) \rightarrow \mathcal{L}(\mathcal{H})$  is a POM if and only if for any unit vector  $\varphi \in \mathcal{H}$ , the map  $X \mapsto p_{\varphi}^E(X) := \langle \varphi | E(X) \varphi \rangle$  is a probability measure. We also recall that a POM  $E : \mathcal{B}(\Omega) \rightarrow \mathcal{L}(\mathcal{H})$  is projection valued, that is,  $E(X)^2 = E(X)$  for all  $X \in \mathcal{B}(\Omega)$ , if and only if it is multiplicative, that is,  $E(X \cap Y) = E(X)E(Y)$  for all  $X, Y \in \mathcal{B}(\Omega)$ . We say that a POM  $E : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$  is *phase shift covariant* if

$$(1) \quad R(\theta)E(X)R(\theta)^* = E(X \oplus \theta)$$

for all  $X \in \mathcal{B}([0, 2\pi))$ ,  $\theta \in \mathbb{R}$ , where  $X \oplus \theta := \{x \in [0, 2\pi) \mid (x - \theta) \bmod 2\pi \in X\}$ , that is, for any unit vector  $\varphi \in \mathcal{H}$

$$p_{R(-\theta)\varphi}^E(X) = p_{\varphi}^E(X \oplus \theta)$$

for all  $X \in \mathcal{B}([0, 2\pi))$ ,  $\theta \in \mathbb{R}$ . We take Eq. (1) to formalize Condition P1.

## 3. COHERENT STATES AND PHASE SHIFT COVARIANT OPERATOR MEASURES

Let  $|z\rangle$  be a coherent state,  $E : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$  a POM, and  $p_{|z\rangle}^E$  the corresponding probability measure. In order that the numbers  $p_{|z\rangle}^E(X)$  could be interpreted as measurement outcome probabilities for the coherent state phase measurements, it is natural to require that they fulfill the following covariance condition:

$$(2) \quad p_{|ze^{-i\theta}\rangle}^E(X) = p_{|z\rangle}^E(X \oplus \theta),$$

for all  $z \in \mathbb{C}$ ,  $\theta \in [0, 2\pi)$ , and  $X \in \mathcal{B}([0, 2\pi))$ , that is, the probability for a phase measurement to give a result in the set  $X$  in the phase shifted coherent state  $R(-\theta)|z\rangle$  is equal to the probability that the measurement in the coherent state  $|z\rangle$  leads to a result in the shifted set  $X \oplus \theta$ . The following result is a well-known consequence of the properties of coherent states (see, e.g. [7, p. 13]). For completeness we give here its simple direct proof.

**Theorem 1.** *Let  $E : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$  be a POM. If for all  $z \in \mathbb{C}$ , the coherent state probability measures  $p_{|z\rangle}^E$  satisfy the covariance condition (2), then  $E$  is phase shift covariant.*

Proof. Let  $A \in \mathcal{L}(\mathcal{H})$ , and denote its matrix elements with respect to the number states as  $A_{n,m} = \langle n|A|m\rangle$ ,  $n, m \in \mathbb{N}$ . Assume that  $\langle z|A|z\rangle = 0$  for all  $z \in \mathbb{C}$ . Since

$$\langle z|A|z\rangle = e^{-|z|^2} \sum_{n,m=0}^{\infty} A_{n,m} \frac{|z|^{n+m}}{\sqrt{n!m!}} e^{i(n-m)\arg z}$$

is a Fourier series with absolute convergence, it follows that the Fourier coefficient

$$f_k(x) := \sum_{n=0}^{\infty} A_{n,n+k} \frac{x^n}{\sqrt{n!(n+k)!}} = 0$$

for all  $x \in [0, \infty)$  and  $k \in \mathbb{N}$ . Due to the uniform convergence,

$$0 = \frac{d^s f_k}{dx^s}(0) = A_{s,s+k} \sqrt{\frac{s!}{(s+k)!}}$$

for all  $s \in \mathbb{N}$ , the matrix elements  $A_{n,m} = 0$  for all  $m \geq n$ . Similarly, one proves that  $A_{n,m} = 0$  for all  $m < n$ , so that  $A = O$ . Since (2) equals the condition

$$\langle z|[E(X \oplus \theta) - R(\theta)E(X)R(\theta)^*]|z\rangle = 0$$

for all  $z \in \mathbb{C}$ ,  $X \in \mathcal{B}([0, 2\pi))$ , and  $\theta \in \mathbb{R}$ , the theorem follows.  $\square$

We call a phase shift covariant normalized positive operator measure  $E : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$  a *covariant phase observable*. Thus, a covariant phase observable is a POM which satisfies Conditions P0 and P1 (or equivalently P0 and Eq. (2)).

#### 4. COVARIANT PHASE OBSERVABLES AND THEIR OPERATORS

A positive semidefinite complex matrix  $(c_{n,m})_{n,m \in \mathbb{N}}$  with  $c_{n,n} = 1$ ,  $n \in \mathbb{N}$ , is called a phase matrix. It is known that  $(c_{n,m})_{n,m \in \mathbb{N}}$  is a phase matrix if and only if  $c_{n,m} = \langle \psi_n | \psi_m \rangle$  for some sequence  $(\psi_n)_{n \in \mathbb{N}} \subset \mathcal{H}$  of unit vectors [8, 9]. Then  $E : \mathcal{B}[0, 2\pi) \rightarrow \mathcal{L}(\mathcal{H})$  is a covariant phase observable if and only if

$$E(X) = \sum_{n,m=0}^{\infty} c_{n,m} \frac{1}{2\pi} \int_X e^{i(n-m)\theta} d\theta |n\rangle \langle m|$$

where  $(c_{n,m})_{n,m \in \mathbb{N}}$  is a phase matrix,  $|n\rangle \langle m|$  is the rank-one operator  $\mathcal{H} \ni \psi \mapsto \langle m | \psi \rangle |n\rangle \in \mathcal{H}$ , and where the series converges in the weak operator topology [10, 11, 9]. It is well known that a covariant phase observable cannot be projection valued. Clearly, the support of a covariant phase observable is the closed phase interval  $[0, 2\pi]$ .

For any covariant phase observable  $E$  the phase distribution in the number state  $|n\rangle$  is uniform. Indeed,  $p_{|n\rangle}^E(X) = \frac{1}{2\pi} \int_X d\theta$  for all  $X \in \mathcal{B}([0, 2\pi))$ . This is a fundamental property of a phase observable. However, it does not force a POM to be covariant under the phase shifts generated by the number observable. In fact, consider the following normalized positive operator measure

$$F : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H}), \quad X \mapsto \frac{1}{2\pi} \int_X d\theta I + \frac{1}{2\pi} \int_X \sin \theta d\theta (|0\rangle \langle 1| + |1\rangle \langle 0|)$$

Clearly, the number state probability measures  $p_{|n\rangle}^F$  are uniform,  $\langle n|F(X)|n\rangle = \frac{1}{2\pi} \int_X d\theta$  for all  $X \in \mathcal{B}([0, 2\pi))$ , but  $F$  is not phase shift covariant. This shows that P0 and P2 do not imply P1.

For any continuous function  $f \in C([0, 2\pi])$  the integral of  $f$  with respect to  $E$  is a (weakly defined) bounded operator denoted as  $\int_0^{2\pi} f(\theta) dE(\theta)$ . In particular, the moment operators of the covariant phase observable  $E$ ,

$$E^{(k)} := \int_0^{2\pi} \theta^k dE(\theta), \quad k \in \mathbb{N},$$

are bounded self-adjoint operators. It is a well-known consequence of the Weierstrass approximation theorem and the uniqueness part of the Riesz representation theorem that the moment operators  $E^{(k)}$ ,  $k \in \mathbb{N}$ , determine  $E$  uniquely. Since  $c_{n,m} = \langle n | E^{(1)} | m \rangle i(n-m)$  for all  $n \neq m$ , the phase observable  $E$ , though not projection valued, is already determined by its first moment operator  $E^{(1)}$ . We emphasize that the spectral measure  $M$  of the moment operator  $E^{(1)}$  is not phase shift covariant. Also, the support of  $M$  is not necessarily the whole interval  $[0, 2\pi]$ , and  $M$  is *never* completely random in the number states. The latter result follows from the fact that if  $M$  was completely random in a number state  $|n\rangle$  then the variance must be  $\pi^2/3$ . The variance is now

$$\left\langle n \left| (E^{(1)})^2 \right| n \right\rangle - (\langle n | E^{(1)} | n \rangle)^2 = \sum_{m=0, m \neq n}^{\infty} \frac{|c_{n,m}|^2}{(n-m)^2} \leq \sum_{s=1}^n \frac{1}{s^2} + \sum_{t=1}^{\infty} \frac{1}{t^2} < \frac{\pi^2}{3}$$

for all  $n \in \mathbb{N}$ . Thus, P1 and P2 (and P0 in many cases) do not hold for  $M$ . Note also that for Wigner-Weyl quantized phase angle P1 and P2 do not hold [3, p. 458].

The cyclic moments of  $E$ ,  $V^{(k)} = \int_0^{2\pi} e^{ik\theta} dE(\theta)$ ,  $k \in \mathbb{N}$ , form another important class of bounded operators constructed from  $E$ . They constitute a nonunitary representation  $k \mapsto V^{(k)}$  of the additive semigroup  $\mathbb{N}$  provided that  $E$  is strong, that is, if  $V^{(k)} = (V^{(1)})^k$  for all  $k \in \mathbb{N}$ . This is a necessary condition for a covariant phase observable to generate number shifts, that is, for the expression of P3.

**4.1. The canonical covariant phase observable.** The constant matrix  $c_{n,m} \equiv 1$ , or, equivalently, a constant sequence  $\psi_n \equiv \psi$  determines the canonical phase observable

$$E_{\text{can}}(X) := \sum_{n,m=0}^{\infty} \frac{1}{2\pi} \int_X e^{i(n-m)\theta} d\theta |n\rangle \langle m|.$$

This is the unique covariant phase observable associated with the polar decomposition of the lowering operator  $a = V|a|$ , where the partial isometry  $V$  is the first cyclic moment  $V^{(1)}$  of  $E_{\text{can}}$  [10]. *The canonical phase  $E_{\text{can}}$  is strong, and it is (up to unitary equivalence) the only covariant phase observable which generates number shifts*, that is,  $V^{(k)}|n+k\rangle = |n\rangle$  for all  $k, n \in \mathbb{N}$  [12]. The unitary equivalence is to be understood in the sense of covariance systems, which, in the present case means that two phase observables  $E_1$  and  $E_2$  are unitarily equivalent if  $E_1 = E_2^U := U E_2 U^*$  for some unitary operator  $U$  diagonalized by the number operator  $N$ . For example,  $E_{\text{can}}^U$  is a phase observable determined by the phase matrix  $(e^{i(v_n - v_m)})_{n,m \in \mathbb{N}}$  where  $(v_n)_{n \in \mathbb{N}} \subset [0, 2\pi)$ . We conclude that  $E_{\text{can}}$  and  $N$  constitute a true canonical pair:  $N$  generates phase shifts and  $E_{\text{can}}$  generates number shifts.

**4.2. Coherent state phase densities and uncertainties.** Let  $g_{|z\rangle}^E$  be the probability density of the coherent state phase probability measure  $p_{|z\rangle}^E$ , that is,  $p_{|z\rangle}^E(X) = (2\pi)^{-1} \int_X g_z^E(\theta) d\theta$ . Since

$$g_{|z\rangle}^E(\theta) = \sum_{n,m=0}^{\infty} c_{n,m} \frac{|z|^{n+m}}{\sqrt{n!m!}} e^{i(n-m)(\theta - \arg z)},$$

and  $c_{n,m} = 1$ , we have

$$g_{|z\rangle}^E(\arg z) \leq g_{|z\rangle}^{E_{\text{can}}}(\arg z)$$

for any covariant phase observable  $E$  showing that the canonical phase gives the highest value for the coherent state phase density at  $\arg z$ . We note also that  $g_{|z\rangle}^{E_{\text{can}}}$  tends to a  $2\pi$ -periodic Dirac  $\delta$ -distribution in the classical limit  $|z| \rightarrow \infty$  [13, 14, 12].

For  $2\pi$ -periodic probability densities  $g$  the standard deviation is not a good measure of uncertainty, the appropriate measure being the Lévy-measure [15, 16]

$$\text{Lévy}(g) := \inf_{\alpha, \beta \in \mathbb{R}} \left\{ \frac{1}{2\pi} \int_{\alpha-\pi}^{\alpha+\pi} (\theta - \beta)^2 g(\theta) d\theta \right\}.$$

For the canonical phase this leads, in coherent states with the amplitude  $|z| > 1/2$ , to the following phase-number uncertainty relation

$$\Delta_{|z\rangle} E_{\text{can}} \Delta_{|z\rangle} N \sim \frac{1}{2},$$

where  $\Delta_{|z\rangle} E_{\text{can}} = \sqrt{\text{Lévy}(g_{|z\rangle}^{E_{\text{can}}})}$  and  $\Delta_{|z\rangle} N = |z|$  (see [12, 14]).

**4.3. Covariant phase space phase observables.** The phase space phase observable generated by a number state  $|s\rangle$ ,  $s \in \mathbb{N}$ , is of the form

$$E_{|s\rangle}(X) := \frac{1}{\pi} \int_X \int_0^\infty D(re^{i\theta}) |s\rangle \langle s| D(re^{i\theta})^* r dr d\theta,$$

where  $D(re^{i\theta}) = \exp[r(e^{i\theta}a^* - e^{-i\theta}a)]$  is the unitary phase space translation operator. For the ground state  $|0\rangle$  this observable is simply

$$\begin{aligned} E_{|0\rangle}(X) &= \frac{1}{\pi} \int_X \int_0^\infty |re^{i\theta}\rangle \langle re^{i\theta}| r dr d\theta \\ &= \sum_{n,m=0}^\infty c_{n,m}^{(0)} \frac{1}{2\pi} \int_X e^{i(n-m)\theta} d\theta |n\rangle \langle m|, \end{aligned}$$

with the matrix elements

$$c_{n,m}^{(0)} := \frac{\Gamma((n+m)/2 + 1)}{\sqrt{n!m!}}$$

(see, e.g. [17]). We note that the angle margin of the Husimi  $Q$ -function  $Q_\varphi(w) = |\langle w|\varphi\rangle|^2$ ,  $w \in \mathbb{C}$ , of a vector state  $\varphi \in \mathcal{H}$ ,  $\|\varphi\| = 1$ , is of the form ( $w = re^{i\theta}$ )

$$\int_0^\infty Q_\varphi(re^{i\theta}) dr^2 = \left\langle \varphi \left| \frac{dE_{|0\rangle}}{d\theta}(\theta) \varphi \right. \right\rangle = g_\varphi^{E_{|0\rangle}}(\theta)$$

where  $\frac{dE_{|0\rangle}}{d\theta}(\theta) = \int_0^\infty |re^{i\theta}\rangle \langle re^{i\theta}| dr^2$  is an operator density of  $E_{|0\rangle}$ . Like  $g_{|z\rangle}^{E_{\text{can}}}$ , the distribution  $g_{|z\rangle}^{E_{|s\rangle}}$  tends to a  $2\pi$ -periodic Dirac  $\delta$ -distribution in the classical limit  $|z| \rightarrow \infty$  and  $\Delta_{|z\rangle} E_{|s\rangle} \Delta_{|z\rangle} N \sim \sqrt{\frac{s+1}{2}}$  (see [12]). As it stands the only phase observable actually measured is  $E_{|0\rangle}$  and it can be measured by using e.g. the double homodyne detection (see, e.g. [17]).

the positive semidefinite form. For  $\psi = \sum_{n \in J} \psi_n |n\rangle$ ,  $\psi_n \in \mathbb{C}$ , let  $\mathcal{M}_J^* := \text{lin}\{|n\rangle : n \in J\}$ , and let  $\mathcal{H}_J := \overline{\mathcal{M}_J}$ , which is a Hilbert subspace of  $\mathcal{H}$ . Let  $\mathcal{M}_J^*$  be the [algebraic] dual of  $\mathcal{M}_J$ . Using the Dirac notation, we thus get  $\sum_{n \in J} t_n \langle n| \in \mathcal{M}_J^*$  for an arbitrary sequence  $(t_n)_{n \in J} \subset \mathbb{C}$ . Here  $\langle n|$  denotes the functional  $\mathcal{H} \ni \varphi \mapsto \langle n|\varphi \rangle \in \mathbb{C}$ . Recall that  $\mathcal{M}_J \subseteq \mathcal{H}_J \simeq \mathcal{H}_J^* \subseteq \mathcal{M}_J^*$  and that the equalities hold if and only if the number of elements of  $J$  is finite ( $\#J < \infty$ ).

We let  $(F|$  denote a generic element of  $\mathcal{M}_J^*$  and we call it a bra vector. For a given bra vector  $(F| \in \mathcal{M}_J^*$  the ket vector  $|F\rangle$  means the antilinear mapping  $\mathcal{M}_J \ni \psi \mapsto \langle \psi|F\rangle := \overline{(F|\psi)} \in \mathbb{C}$ . For a complex matrix  $(d_{n,m})_{n,m \in J}$  the formal double series  $\sum_{n,m \in J} d_{n,m} |n\rangle \langle m|$  is to be understood as the following sesquilinear form:

$$\mathcal{M}_J \times \mathcal{M}_J \ni (\varphi, \psi) \mapsto D(\varphi, \psi) := \sum_{n,m \in J} d_{n,m} \langle \varphi|n\rangle \langle m|\psi\rangle \in \mathbb{C}.$$

Similarly, we let  $|F\rangle(F|$  denote the sesquilinear form  $\mathcal{M}_J \times \mathcal{M}_J \ni (\varphi, \psi) \mapsto \langle \varphi|F\rangle (F|\psi) \in \mathbb{C}$ .

By definition, matrix  $(d_{n,m})_{n,m \in J}$  is positive semidefinite if and only if  $D(\psi, \psi) \geq 0$  for all  $\psi \in \mathcal{M}_J$ . We recall that if, in addition,  $J = \mathbb{N}$  and  $d_{n,n} = 1$  for all  $n \in \mathbb{N}$ , the matrix  $(d_{n,m})_{n,m \in \mathbb{N}}$  is a phase matrix.

**Theorem 2.** *The following statements are equivalent:*

- (i)  $(d_{n,m})_{n,m \in J}$  is positive semidefinite;
- (ii)  $d_{n,m} = \langle \psi_n|\psi_m\rangle$ ,  $n, m \in J$ , for some vector sequence  $(\psi_n)_{n \in J} \subset \mathcal{H}_J$ ;
- (iii)  $d_{n,m} = \sum_{k \in J} \langle n|F_k\rangle (F_k|m\rangle)$ ,  $(F_k| \in \mathcal{M}_J^*$ ,  $k \in J$ , and  $\sum_{k \in J} |(F_k|n)|^2 < \infty$  for all  $n, m \in J$ .

*Proof.* It is well-known that (i) equals (ii), see, e.g. [8, Chpt 3]. Let  $(\psi_n)_{n \in J}$  be a sequence of vectors in  $\mathcal{H}_J$ , and put, for all  $n, m \in J$ ,  $d_{n,m} = \langle \psi_n|\psi_m\rangle$ . Then

$$d_{n,m} = \langle \psi_n|\psi_m\rangle = \sum_{k \in J} \langle \psi_n|k\rangle \langle k|\psi_m\rangle = \sum_{k \in J} \langle n|F_k\rangle (F_k|m\rangle),$$

where  $(F_k| := \sum_{n \in J} \langle k|\psi_n\rangle \langle n|$  and  $\sum_{k \in J} |(F_k|n)|^2 = \langle \psi_n|\psi_n\rangle < \infty$ .

Suppose then that  $(F_k| \in \mathcal{M}_J^*$  and  $\sum_{k \in J} |(F_k|n)|^2 < \infty$  for all  $k, n \in J$ . Then, by the Cauchy-Schwarz inequality [for  $\ell_{\mathbb{C}}^2(J)$ ], the series  $\sum_{k \in J} \langle \varphi|F_k\rangle (F_k|\psi\rangle$  converges for all  $\varphi, \psi \in \mathcal{M}_J$  and it is nonnegative when  $\varphi = \psi$ . Defining, for all  $n, m \in J$ ,  $d_{n,m} = \sum_{k \in J} \langle n|F_k\rangle (F_k|m\rangle$ , we see that  $D(\varphi, \psi) = \sum_{n,m \in J} d_{n,m} \langle \varphi|n\rangle \langle m|\psi\rangle = \sum_{k \in J} \langle \varphi|F_k\rangle (F_k|\psi\rangle$  is a positive sesquilinear form, that is,  $(d_{n,m})_{n,m \in J}$  is positive semidefinite.  $\square$

For a positive semidefinite matrix  $(c_{n,m})_{n,m \in J}$  with  $c_{n,n} \equiv 1$  one gets

$$\sum_{n,m \in J} c_{n,m} |n\rangle \langle m| = \sum_{k \in J} |F_k\rangle (F_k|,$$

where now  $(F_k| \in \mathcal{M}_J^*$  and  $\sup_{k \in \mathbb{N}} \{ |(F_k|n)| : n \in J \} \leq 1$  for all  $k \in J$ . For the canonical phase, one may choose  $\psi_n \equiv |0\rangle$ , and thus  $|F_0\rangle = \sum_{n=0}^{\infty} |n\rangle$  and  $|F_k\rangle = 0$  otherwise. Note that  $R(\theta)|F_0\rangle \equiv |\theta\rangle := \sum_{n=0}^{\infty} e^{in\theta} |n\rangle$  and

$$E_{\text{can}}(X) = \frac{1}{2\pi} \int_X |\theta\rangle \langle \theta| d\theta.$$

For the trivial phase  $(c_{n,m} \equiv \delta_{n,m})$  one gets  $|F_k\rangle = |k\rangle$ ,  $\sum_{k=0}^{\infty} |F_k\rangle (F_k| = I$ , and  $E_{\text{triv}}(X) = (2\pi)^{-1} \int_X d\theta I$ .

**Proposition 1.** *With the above notations,  $\sum_{n,m \in J} c_{n,m} |n\rangle \langle m| = |F\rangle (F|$  for some  $(F| \in \mathcal{M}_J^*$  if and only if  $c_{n,m} = e^{i(v_n - v_m)}$  for all  $n, m \in J$ .*

1983]. If  $\sum_{n,m \in J} c_{n,m} |v_n\rangle\langle v_m| = |F\rangle\langle F|$  for all  $n \in J$ , and thus  $|F\rangle = \sum_{n \in J} e^{iv_n} |n\rangle$ . Hence  $c_{n,m} = e^{i(v_n - v_m)}$  for all  $n, m \in J$ . Conversely, if  $c_{n,m} = e^{i(v_n - v_m)}$  for all  $n, m \in J$ , then  $\sum_{n,m \in J} c_{n,m} |n\rangle\langle m| = |F\rangle\langle F|$ , with  $|F\rangle = \sum_{n \in J} e^{iv_n} |n\rangle$ .  $\square$

From Proposition 1 one sees that the canonical phase observable is (up to unitary equivalence) the only phase observable which can be written in the form

$$X \mapsto \frac{1}{2\pi} \int_X R(\theta) |F\rangle\langle F| R(\theta)^* d\theta$$

using a single generalized state  $|F\rangle$ .

## 5. THE PEGG-BARNETT PHASE AND ITS GENERALIZATION

For a fixed  $s \in \mathbb{N}$ , define  $\mathcal{H}_s := \text{lin}\{|n\rangle \mid n = 0, 1, \dots, s\} \subset \mathcal{H}$ , and let  $\theta_{s,k} := k \frac{2\pi}{s+1}$ ,  $k = 0, 1, \dots, s$ . The Pegg-Barnett (PB) phase states are then

$$|\theta_{s,k}\rangle := \frac{1}{\sqrt{s+1}} \sum_{n=0}^s e^{in\theta_{s,k}} |n\rangle,$$

and they determine a spectral measure

$$\begin{aligned} E_s^{\text{can}}([a, b)) &:= \sum_{\theta_{s,k} \in [a, b)} |\theta_{s,k}\rangle\langle\theta_{s,k}| \quad (0 \leq a < b \leq 2\pi) \\ &= \sum_{n,m=0}^s \underbrace{\frac{1}{s+1} \sum_{\theta_{s,k} \in [a, b)} e^{i(n-m)\theta_{s,k}}}_{\rightarrow (2\pi)^{-1} \int_{[a, b)} e^{i(n-m)\theta} d\theta} |n\rangle\langle m|. \end{aligned}$$

Since  $\|E_s^{\text{can}}([a, b))\| \leq 1$  for all  $s \in \mathbb{N}$  it follows that

$$(3) \quad \text{w-lim}_{s \rightarrow \infty} E_s^{\text{can}}([a, b)) = E_{\text{can}}([a, b)).$$

(see also [18]).

Discrete phase shifts  $\{\theta_{s,k} \mid k = 0, 1, \dots, s\}$  with the addition modulo  $2\pi$  form a group which is isomorphic to a finite additive group  $\mathbb{Z}_{s+1}$  [or a cyclic group  $C_{s+1}$ ]. Consider the following unitary representation of the group  $\mathbb{Z}_{s+1}$  of discrete phase shifts:

$$\mathbb{Z}_{s+1} \ni k + (s+1)\mathbb{Z} \mapsto R(\theta_{s,k}) \in \mathcal{L}(\mathcal{H}).$$

This is well defined since  $R(\theta_{s,k}) = R(\theta_{s,k+(s+1)n})$  for all  $n \in \mathbb{Z}$ . Since  $R(\theta_{s,k})|\theta_{s,l}\rangle = |\theta_{s,k+l}\rangle$  one may easily confirm that

$$R(\theta_{s,k}) E_s^{\text{can}}(\{\theta_{s,l}\}) R(\theta_{s,k})^* = E_s^{\text{can}}(\{\theta_{s,k+l}\})$$

for all  $k, l = 0, \dots, s$ , that is,  $E_s^{\text{can}}$  is (discrete) phase shift covariant. We go on to determine all such POMs.

Let  $J \subseteq \mathbb{N}$ ,  $J \neq \emptyset$ ,  $\mathcal{H}_J = \overline{\mathcal{M}_J}$ , and let  $\mathcal{L}(\mathcal{H}_J)$  denote the set of bounded operators on  $\mathcal{H}_J$ . Fix an  $s \in \mathbb{N}$ , and let  $\mathcal{P}(D_s)$  denote the power set of the set  $D_s := \{\theta_{s,k} \in [0, 2\pi) \mid k = 0, 1, \dots, s\}$ .

**Theorem 3.** *With the above notations, the map*

$$\mathcal{P}(D_s) \ni X \mapsto E_{J,s}(X) := \sum_{\theta_{s,l} \in X} E_{J,s}(\{\theta_{s,l}\}) \in \mathcal{L}(\mathcal{H}_J)$$

*is a normalized positive operator measure satisfying the covariance condition*

$$R(\theta_{s,k}) E_{J,s}(\{\theta_{s,l}\}) R(\theta_{s,k})^* = E_{J,s}(\{\theta_{s,k+l}\}), \quad k, l = 0, 1, \dots, s,$$

$$E_{J,s}(\{\theta_{s,l}\}) = \frac{1}{s+1} R(\theta_{s,l}) A^J R(\theta_{s,l})^*, \quad k = 0, \dots, s,$$

where  $A^J$  is a bounded positive operator on  $\mathcal{H}_J$ , with  $\langle n|A^J|n\rangle = 1$  for all  $n \in J$ , and, for any  $n \neq m \in J$ , if  $\langle n|A^J|m\rangle \neq 0$ , then  $|n-m| \notin (s+1)\mathbb{Z}^+$ . The covariant POM  $E_{J,s}$  is projection valued if and only if  $\langle n|A^J|m\rangle = e^{i(v_n-v_m)}$  for all  $n, m \in J$ , where  $(v_n)_{n \in J} \subset [0, 2\pi)$ , and  $\#J = s+1$ .

*Proof.* Suppose that  $E : \mathcal{P}(D_s) \rightarrow \mathcal{L}(\mathcal{H}_J)$  is a POM for which  $R(\theta_{s,k})E_{J,s}(\{\theta_{s,l}\})R(\theta_{s,k})^* = E_{J,s}(\{\theta_{s,k+l}\})$ . Choosing  $l = 0$  one sees that  $E_{J,s}(\{\theta_{s,k}\}) = R(\theta_{s,k})E_{J,s}(\{0\})R(\theta_{s,k})^*$  where  $E_{J,s}(\{0\}) \in \mathcal{L}(\mathcal{H}_J)$  and  $E_{J,s}(\{0\}) \geq 0$ .

Conversely, if  $E_{J,s}(X) = \sum_{k \in X} R(\theta_{s,k})BR(\theta_{s,k})^*$ ,  $X \in \mathcal{P}(D_s)$ , for some  $B \in \mathcal{L}(\mathcal{H}_J)$ ,  $B \geq 0$ , then  $E_{J,s}$  is a covariant positive operator measure. The normalization condition

$$\sum_{k=0}^s R(\theta_{s,k})BR(\theta_{s,k})^* = I|_{\mathcal{H}_J}$$

equals the condition

$$\langle n|B|m\rangle \underbrace{\sum_{k=0}^s e^{2\pi i(n-m)k/(s+1)}}_{=(s+1)\delta_{n-m,(s+1)t}, \quad t \in \mathbb{Z}} = \delta_{n,m}, \quad n, m \in J,$$

which equals the following two conditions:

- (i) for  $n \neq m$ , if  $\langle n|B|m\rangle \neq 0$  then  $|n-m| \notin (s+1)\mathbb{Z}^+$ ,
- (ii)  $\langle n|B|n\rangle = 1/(s+1)$  for all  $n \in J$ .

Thus, if  $B$  is as above and satisfies conditions (i) and (ii), then define  $A^J := (s+1)B$  and the first part of Theorem is proved.

Let  $A_{n,m}^J = \langle n|A^J|m\rangle$  for all  $n, m \in J$ . If  $A_{n,m}^J = e^{i(v_n-v_m)}$ ,  $n, m \in J$ , where  $(v_n)_{n \in J} \subset [0, 2\pi)$  and  $\#J = s+1$ , then it is easy to confirm that  $E_{s,J}$  is a projection measure. Conversely, suppose that  $E_{s,J}$  is a covariant normalized projection measure, that is,

$$E_{s,J}(\{\theta_{s,k}\})E_{s,J}(\{\theta_{s,l}\}) = \delta_{k,l}E_{s,J}(\{\theta_{s,k}\})$$

for all  $k, l \in \{0, 1, \dots, s\}$ . By direct calculation, one sees that this equals the fact that

$$(4) \quad \frac{1}{s+1} \sum_{t \in J} A_{n,t}^J A_{t,m}^J e^{2\pi i[k(n-t)+l(t-m)]/(s+1)} = \delta_{k,l} A_{n,m}^J e^{2\pi i(n-m)k/(s+1)}$$

for all  $n, m \in J$  and  $k, l \in \{0, 1, \dots, s\}$ . Multiply both sides of this equation by  $e^{-2\pi i k q/(s+1)}/(s+1)$  and sum up with respect to  $k$  to get

$$A_{n,n-q+u(s+1)}^J A_{n-q+u(s+1),m}^J = A_{n,m}^J,$$

which holds for all  $n, m \in J$  and for all  $q, u \in \mathbb{Z}$  for which  $n-q+u(s+1) \in J$ . Substituting  $v := n-q+u(s+1)$  one gets

$$(5) \quad A_{n,v}^J A_{v,m}^J = A_{n,m}^J$$

for all  $n, m, v \in J$ . Use Eq. (5) in Eq. (4) to get

$$(6) \quad \sum_{t \in J} e^{2\pi i t(l-k)/(s+1)} = (s+1)\delta_{k,l}.$$

From  $l = k$  we see that  $\#J = s+1$ . If  $\#J = s+1$  then Eq. (6) clearly holds.



Since the operator  $\sum_{n,m \in J} |n\rangle\langle m|$  is positive and  $A_{n,n}^J = 1$  then  $|A_{n,m}^J| \leq 1$  for all  $n, m \in J$ . Using Eq. (4) with  $n = m$  and  $k = l$  and the fact that  $\#J = s + 1$  it follows that  $|A_{n,m}^J| = 1$  for all  $n, m \in J$ . Due to the positiveness, if  $\#J \geq 3$  then

$$\begin{vmatrix} 1 & A_{i,j}^J & A_{i,k}^J \\ \overline{A_{i,j}^J} & 1 & A_{j,k}^J \\ \overline{A_{i,k}^J} & \overline{A_{j,k}^J} & 1 \end{vmatrix} \geq 0$$

for all  $i, j, k \in J$  for which  $i < j < k$ . Using this and the condition  $|A_{n,m}^J| \equiv 1$  one gets by direct calculation that  $A_{n,m}^J = e^{i(v_n - v_m)}$ ,  $n, m \in J$ , where  $(v_n)_{n \in J} \subset [0, 2\pi)$ . If  $\#J = 2$  or  $\#J = 1$  this holds trivially.  $\square$

Since  $A^J$  is bounded, we may now write

$$E_{J,s}(\{\theta_{s,l}\}) = \sum_{n,m \in J} \frac{1}{s+1} A_{n,m}^J e^{i(n-m)\theta_{s,l}} |n\rangle\langle m|,$$

where  $A_{n,m}^J = \langle n|A|m\rangle$ . Therefore, if  $\#J = \infty$ , there is no bounded operator  $A$  such that  $A_{n,m}^J = e^{i(v_n - v_m)}$  for all  $n, m \in J$ . In particular, the case  $A_{n,m}^J \equiv 1$  requires  $J$  to be finite.

Let  $E$  be a covariant phase observable with the phase matrix  $(c_{n,m})_{n,m \in \mathbb{N}}$ . Let  $J \equiv J(s) := \{0, 1, \dots, s\}$ . If  $\lim_{s \rightarrow \infty} A_{n,m}^{J(s)} = c_{n,m}$  for all  $n, m \in \mathbb{N}$  then  $\text{w-lim}_{s \rightarrow \infty} E_{J(s),s}([a, b]) = E([a, b])$ . The sequence  $(E_{J(s),s})_{s \in \mathbb{N}}$  gives an approximation sequence for  $E$ . In this way we have generalized the PB-formalism to cover all covariant phase observables. Indeed, for a given phase matrix  $(c_{n,m})_{n,m \in \mathbb{N}}$  one may choose, for example,  $A_{n,m}^{J(s)} = c_{n,m}$  for  $n, m \leq s$  to generalize (3). Especially, the operators  $E_{|0\rangle}([a, b])$  of the phase observable  $E_{|0\rangle}$  can be approximated weakly by the sequence of operators

$$\sum_{n,m=0}^s \frac{\Gamma((n+m)/2 + 1)}{\sqrt{n!m!}} \frac{1}{s+1} \sum_{\theta_{s,k} \in [a,b]} e^{i(n-m)\theta_{s,k}} |n\rangle\langle m|.$$

when  $s \rightarrow \infty$ .

The approximation sequence  $(E_{J(s),s})_{s \in \mathbb{N}}$  of a phase observable  $E$  gives a discretization of  $E$  when  $\text{w-lim}_{s \rightarrow \infty} E_{J(s),s}([a, b]) = E([a, b])$ , and it satisfies the spectral accuracy condition [3, p. 495]

$$\sup_{\theta \in [a,b]} \left( \min_{\lambda \in D_s} |\theta - \lambda| \right) \rightarrow 0$$

for all  $a < b \in [0, 2\pi]$  when  $s \rightarrow \infty$ .

Theorem 3 shows that only  $E_{\text{can}}^U$  has a projection valued discretization. In this sense one may say that  $E_{\text{can}}^U$  is almost projection valued. This is somewhat striking since  $E_{\text{can}}^U$  is known to be totally noncommutative [19]. Also, from Proposition 1 one obtains that  $E_{\text{can}}$  is (up to unitary equivalence) the only phase observable which is determined by only one (generalized) phase state  $|\theta\rangle$ , and it is (up to unitary equivalence) the only phase observable which has the approximation sequence determined by one discrete phase state  $|\theta_{s,k}\rangle$ . That is,  $E_{\text{can}}^U$  can be approximated by the operator measures of the form

$$\frac{1}{s+1} \sum_{\theta_{s,k} \in [a,b]} R(\theta_{s,k})|F\rangle\langle F|R(\theta_{s,k})^*.$$

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